

# “Velocities” in Quantum Mechanics

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## Abstract

The present paper deals with some kind of quantum “velocity” which is introduced by the method of hydrodynamical analogy. It is found that this “velocity” is in general irrotational, namely, a vorticity vanishes, and then a velocity potential must exist in quantum mechanics. In some elementary examples of stable systems we will see what the “velocities” are. In particular, the two-dimensional flows of these examples can be expressed by complex velocity potentials whose real and imaginary parts are the velocity potentials and stream functions, respectively.

It is well known that the probability density  $\rho(t, \mathbf{r})$  and the probability current  $\mathbf{j}(t, \mathbf{r})$  of a state  $\psi(t, \mathbf{r})$  in non-relativistic quantum mechanics are given by [1, 2, 3]

$$\rho(t, \mathbf{r}) \equiv |\psi(t, \mathbf{r})|^2, \quad (1)$$

$$\mathbf{j}(t, \mathbf{r}) \equiv \Re [\psi(t, \mathbf{r})^* (-i\hbar \nabla) \psi(t, \mathbf{r})] / m, \quad (2)$$

where  $m$  is the mass of the particle. They satisfy the equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (3)$$

However, these  $\rho$  and  $\mathbf{j}$  are independent of the time  $t$  when considering state is a closed state of the stable system (e.g. the harmonic oscillator, the hydrogen atom) or a stationary state (e.g. the free particle). For such closed stationary states the probability currents will be at rest or uniform, and the equation (3) reduces to

$$\nabla \cdot \mathbf{j} = 0.$$

Thus the probability currents are simple for the closed stationary states of the stable system. The present paper deals with some kind of “*velocity*”, which is different from the eigenvalue of the velocity operator and which is a stepping-stone to a more description in quantum mechanics.

We start out with the equations of hydrodynamics, consisting of Euler’s equation of continuity for the density and velocity of fluid and so on. Let us try to introduce a “velocity” which will be the analogue of the hydrodynamical one. In hydrodynamics [4, 5, 6], the fluid at one time can be represented by the density  $\rho$  and the fluid velocity  $\mathbf{v}$ . They satisfy Euler’s equation of continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (4)$$

Comparing this equation with (3), we are thus led to the following *definition for the quantum “velocity” of a state  $\psi(t, \mathbf{r})$* ,

$$\text{“}\mathbf{v}\text{”} \equiv \frac{\mathbf{j}(t, \mathbf{r})}{|\psi(t, \mathbf{r})|^2}, \quad (5)$$

in which  $\mathbf{j}(t, \mathbf{r})$  is given by (2). It is stressed that this “velocity” is different from the eigenvalue of the velocity operator  $\hat{\mathbf{v}} = -i\hbar \nabla / m$ , except for the cases that  $\psi(t, \mathbf{r})$  is an eigenfunction of the momentum operator  $\hat{\mathbf{p}} = -i\hbar \nabla$ . Note that, if we can separate variables of  $\psi(t, \mathbf{r})$ , then “ $\mathbf{v}$ ” does not contain the time  $t$  explicitly.

Equation (5) is justified from the following point of view. In semiclassical cases the time-dependent wave function can be written [1, 2, 3]

$$\psi(t, \mathbf{r}) = \sqrt{\rho} e^{iS/\hbar}, \quad (6)$$

where  $\rho$  is the probability density (1) and  $S$  is the quantum analogue of the classical action which is a real function of the  $t$  and  $\mathbf{r}$ . The “velocity” (5) gives

$$“\mathbf{v}” = \nabla S/m. \quad (7)$$

The right-hand side is known just as the velocity in classical dynamics. Sakurai [3] introduced the “velocity” by this relation (7).

We shall now consider the *vorticity* in quantum mechanics. It is defined as in the hydrodynamics, by

$$\boldsymbol{\omega} \equiv \nabla \times “\mathbf{v}”. \quad (8)$$

With the above definition we have the next theorem.

**THEOREM.** *If the variables  $t$ ,  $\mathbf{r}$  of a time-dependent wave function  $\psi(t, \mathbf{r})$  can be separated, then*

$$\boldsymbol{\omega} = 0 \quad (9)$$

*for the domain in which the “velocity” does not have singularities.*

To prove the theorem, it is sufficient to evaluate  $\omega$  in the two-dimensional  $xy$  space. From the hypothesis, the time-dependent wave function must be of the form

$$\psi(t, x, y) = T(t)X(x)Y(y).$$

The components of (2) now give

$$\begin{aligned} j_x(t, x, y) &= |T(t)|^2 |Y(y)|^2 \Re [X(x)^*(-i\hbar)X(x)'] / m, \\ j_y(t, x, y) &= |T(t)|^2 |X(x)|^2 \Re [Y(y)^*(-i\hbar)Y(y)'] / m. \end{aligned}$$

The components of (5) then give

$$\begin{aligned} “v_x” &= \Re [X(x)^*(-i\hbar)X(x)'] / m |X(x)|^2, \\ “v_y” &= \Re [Y(y)^*(-i\hbar)Y(y)'] / m |Y(y)|^2, \end{aligned}$$

showing that each component of “velocity” is depending only on the corresponding Cartesian coordinate. When this “velocity” is not singular, the result of its rotation is

$$\omega = \frac{\partial “v_y”}{\partial x} - \frac{\partial “v_x”}{\partial y} = 0.$$

Thus the theorem is proved in two-dimensional space.

Theorem is still valid in three-dimensional space or in terms of polar coordinates, only formal changes being needed in the proof.

In hydrodynamics [4, 5, 6], the flow satisfying  $\boldsymbol{\omega} = 0$  is called *potential flow* or *irrotational flow*. Thus this theorem asserts that *the flow in quantum mechanics is in general irrotational flow, when all variables are separable*. The “velocity” in irrotational flow satisfying (9) may be described by the gradient of the *velocity potential*  $\Phi$ ,

$$\text{“}\boldsymbol{v}\text{”} = \nabla\Phi. \quad (10)$$

Comparing this with (7), we see that

$$\Phi = S/m, \quad (11)$$

in semiclassical cases. The right-hand side here is undefined to the extent of an arbitrary additive real constant.

We now proceed to study only the *two-dimensional* or *plane flow*. Let us consider the “velocity” (5) which is solenoidal, namely

$$\nabla \cdot \text{“}\boldsymbol{v}\text{”} \equiv \frac{\partial \text{“}v_x\text{”}}{\partial x} + \frac{\partial \text{“}v_y\text{”}}{\partial y} = 0. \quad (12)$$

The “velocity” in two-dimensional flow satisfying (12) may be described by the rotation of the *stream function*  $\Psi$ ,

$$\text{“}v_x\text{”} = \frac{\partial \Psi}{\partial y}, \quad \text{“}v_y\text{”} = -\frac{\partial \Psi}{\partial x}. \quad (13)$$

Further, in the two-dimensional irrotational flow, by combining (13) with (10) we get

$$\begin{aligned} \text{“}v_x\text{”} &= \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \\ \text{“}v_y\text{”} &= \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x}. \end{aligned}$$

It is known as *Cauchy-Riemann’s equations* between the velocity potential and the stream function. We can therefore take the *complex velocity potential*

$$W(z) = \Phi(x, y) + i\Psi(x, y), \quad (14)$$

which is a regular function of the complex variable  $z = x + iy$ . The differentiation of  $W(z)$  gives us the *complex velocity*

$$\frac{dW}{dz} = \text{“}v_x\text{”} - i\text{“}v_y\text{”}. \quad (15)$$

In this way we know that in the two-dimensional irrotational flow it is advantageous to use the theory of functions of a complex variable [4, 5, 6].

The solenoidal condition (12) holds for incompressible fluids in hydrodynamics, since  $\rho$  is a constant. In quantum mechanics, however, the probability density (1) depends generally on  $\boldsymbol{r}$ , so that the “velocity” (5) is not always solenoidal.

We shall illustrate the physical value of “velocities” by applying them to examine some elementary examples of stable systems.

**Example 1. The free particle**

Let us first consider the free particle in two dimensions as an example of a stationary state. The plane wave is of the form

$$u_{p_x p_y}(x, y) = a e^{i(p_x x + p_y y)/\hbar}, \quad (16)$$

where  $a$  is independent of  $t$ ,  $x$  and  $y$ . The probability current of the plane wave is

$$j_x(x, y) = |a|^2 p_x / m, \quad j_y(x, y) = |a|^2 p_y / m,$$

and hence their “velocity” is

$$“v_x” = p_x / m, \quad “v_y” = p_y / m.$$

Equations (9) and (12) are easily seen to hold for the plane wave. Therefore the velocity potential satisfying (10) or (11) of the plane wave is

$$\Phi = (p_x x + p_y y) / m, \quad (17)$$

and the stream function satisfying (13) is

$$\Psi = (p_x y - p_y x) / m. \quad (18)$$

For the state represented by (16), the complex velocity potential (14) gives, from (17) and (18)

$$\begin{aligned} W &= (p_x x + p_y y) / m + i(p_x y - p_y x) / m \\ &= (p_x - i p_y) z / m. \end{aligned} \quad (19)$$

According to hydrodynamics [4, 5, 6], the flow round the angle  $\pi/n$  is expressed by the complex velocity potential

$$W = A z^n, \quad (20)$$

$A$  being a number. Equation (19) is of the form (20) with  $n = 1$ , and it shows that *the complex velocity potential of the plane wave just expresses uniform flow.*

**Example 2. The harmonic oscillator**

We shall now consider the eigenstate of the two-dimensional harmonic oscillator as an example of a closed state of the stable system. The eigenfunction is,<sup>1</sup> in terms of the Cartesian coordinates  $x$ ,  $y$ ,

$$u_{n_x n_y}(x, y) = N_{n_x} N_{n_y} e^{-\alpha^2(x^2 + y^2)/2} H_{n_x}(\alpha x) H_{n_y}(\alpha y) \quad \left( \alpha \equiv \sqrt{m\omega/\hbar} \right), \quad (21)$$

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<sup>1</sup>The  $\omega$  here, denoting the angular frequency, is, of course, to be distinguished from the  $\omega$  denoting the vorticity.

where  $N_{n_x}$ ,  $N_{n_y}$  are the normalizing constants. But the Hermite polynomials  $H_{n_x}(\alpha x)$ ,  $H_{n_y}(\alpha y)$  are real functions of  $x$ ,  $y$ , respectively. Thus the probability current of the harmonic oscillator vanishes and their “velocity” also vanishes. Therefore the velocity potential and the stream function all vanish. For the state represented by (21), the complex velocity potential gives

$$W = 0, \quad (22)$$

which expresses *fluid at rest* in hydrodynamics. This fluid at rest, however, is not the only one that is physically permissible for a closed state in quantum mechanics, as we can also have flows which are *vortical*. For these flows the vorticity may contain singularities in the  $xy$ -plane. Such flows will be dealt with in Example 3.

### Example 3. Flows in a central field of force

As a final example we shall consider the bound state in a certain central field of force. The eigenfunction is, in terms of the polar coordinates  $r$ ,  $\theta$ ,  $\varphi$ ,

$$u_{nlm_l}(r, \theta, \varphi) = R_{nl}(r)Y_{lm_l}(\theta, \varphi), \quad (23)$$

where the spherical harmonics  $Y_{lm_l}(\theta, \varphi)$  are of the form

$$Y_{lm_l}(\theta, \varphi) = C_{lm_l}P_l^{|m_l|}(\cos \theta)e^{im_l\varphi}, \quad (24)$$

and  $C_{lm_l}$  are the normalizing constants. But  $R_{nl}(r)$  for the bound state and the associated Legendre polynomials  $P_l^{|m_l|}(\cos \theta)$  are real functions. The polar coordinates  $j_r$ ,  $j_\theta$ ,  $j_\varphi$  of (2) in a central field of force are thus

$$j_r(r, \theta, \varphi) = j_\theta(r, \theta, \varphi) = 0, \quad j_\varphi(r, \theta, \varphi) = |u_{nlm_l}(r, \theta, \varphi)|^2 \frac{m_l \hbar}{mr \sin \theta}.$$

In consequence, a simple treatment becomes possible, namely, we may consider the “velocity” for a definite direction  $\theta$  and then we can introduce the radius  $\rho = r \sin \theta$  in above equations and get a problem in two degrees of freedom  $\rho$ ,  $\varphi$ . The two-dimensional polar coordinates “ $v_\rho$ ”, “ $v_\varphi$ ” of (5) give

$$“v_\rho” = 0, \quad “v_\varphi” = \frac{m_l \hbar}{m\rho}.$$

The divergence of them readily vanishes. If we transform to two-dimensional polar coordinates  $\rho$ ,  $\varphi$ , equations (13) become

$$“v_\rho” = \frac{1}{\rho} \frac{\partial \Psi}{\partial \varphi}, \quad “v_\varphi” = -\frac{\partial \Psi}{\partial \rho}, \quad (25)$$

and the stream function in a central field of force is thus

$$\Psi = -\frac{m_l \hbar}{m} \log \rho. \quad (26)$$

On the other hand, the vorticity (8) satisfies, with the help of (25),

$$\omega = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho "v_\varphi") - \frac{1}{\rho} \frac{\partial}{\partial \varphi} "v_\rho" = -\nabla^2 \Psi, \quad (27)$$

where  $\nabla^2$  is written for the two-dimensional Laplacian operator

$$\nabla^2 \equiv \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2}{\partial \varphi^2}.$$

On substituting (26) in (27) we obtain

$$\omega = \frac{m_l \hbar}{m} \nabla^2 \log \rho = 2\pi \frac{m_l \hbar}{m} \delta(\boldsymbol{\rho}), \quad (28)$$

where  $\delta(\boldsymbol{\rho})$  is the two-dimensional Dirac  $\delta$  function. Thus the vorticity in a central field of force vanishes everywhere except the origin  $\rho = 0$ . This singularity will lie along the quantization axis  $\theta = 0$  and  $\pi$  in three-dimensional space. The velocity potential satisfying (10) or (11) in a central field of force is

$$\Phi = \frac{m_l \hbar}{m} \varphi. \quad (29)$$

For the state represented by (23), the complex velocity potential (14) gives, from (29) and (26)

$$\begin{aligned} W &= \frac{m_l \hbar}{m} \varphi - i \frac{m_l \hbar}{m} \log \rho \\ &= -i \frac{m_l \hbar}{m} \log z, \end{aligned} \quad (30)$$

since  $z = \rho e^{i\varphi}$ . According to hydrodynamics [6], this expresses the *vortex filament*. The strength of the vortex filament is defined by the *circulation* round a closed contour  $C$  encircling the singularity at the origin  $\rho = 0$

$$\Gamma \equiv \oint_C "v_\varphi" ds. \quad (31)$$

We make use of Stokes' theorem,

$$\Gamma = \iint_S \omega dS, \quad (32)$$

where  $S$  is a two-dimensional surface whose boundary is the closed contour  $C$ . On substituting (28) in (32) we obtain

$$\Gamma = 2\pi \frac{m_l \hbar}{m}, \quad (33)$$

where the eigenvalue  $m_l$  of a component of the angular momentum is an integer. Equation (33) informs us that *the circulations are quantized in units of  $2\pi\hbar/m$  for the state (23) moving in a central field of force*. Equation (33) is known as *Onsager's Quantization of Circulations*, in superfluidity [7, 8].

The above examples show the great superiority of the “velocities” in dealing with the flows in quantum mechanics. In particular, the two-dimensional quantum flows can be expressed by complex velocity potentials and their analytical properties. Up to the present we have considered only stable systems. For a non-stationary state of the unstable system the probability density (1) and the probability current (2) are not simple, i.e. they generally depend on the time  $t$  and the coordinate  $\mathbf{r}$ . However, the work will be simple for such unstable systems, since the “velocity” (5) does not involve  $t$ . In fact, for the two-dimensional parabolic potential barrier [9], as an example of the unstable systems, there is the flow round a right angle that is expressed by the complex velocity potential (20) with  $n = 2$ .

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